

# Extended Supersymmetries and the Dirac Operator

A. Kirchberg, J.D. Länge and A. Wipf\*

Theoretisch–Physikalisches Institut,  
Friedrich–Schiller–Universität Jena,  
07743 Jena, Germany

**Abstract:** We consider supersymmetric quantum mechanical systems in arbitrary dimensions on curved spaces with nontrivial gauge fields. The square of the Dirac operator serves as Hamiltonian. We derive a relation between the number of supercharges that exist and restrictions on the geometry of the underlying spaces as well as the admissible gauge field configurations. From the superalgebra with two or more real supercharges we infer the existence of integrability conditions and obtain a corresponding superpotential. This potential can be used to deform the supercharges and to determine zero modes of the Dirac operator. The general results are applied to the Kähler spaces  $\mathbb{CP}^n$ .

**Keywords:** Supersymmetry, Dirac Operator, Complex Manifolds, Kähler Manifolds, Projective Spaces.

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## 1 Introduction

Supersymmetry is a crucial ingredient in many attempts to unify the interactions contained in the standard model of particle physics. It softens the ultraviolet divergences and offers the hope of resolving the hierarchy problem. It arises naturally in low-energy

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\*A.Kirchberg@tpi.uni-jena.de, J.D.Laenge@tpi.uni-jena.de, A.Wipf@tpi.uni-jena.de

limits of string theory. Supersymmetric models are easier to solve than their non-supersymmetric counterparts, since they are more strongly constrained by the higher degree of symmetries.

In recent years we have seen a renewed interest in nonperturbative aspects of strongly interacting supersymmetric theories. This is mainly due to the *Seiberg-Witten solution* for the low-energy effective action of  $\mathcal{N} = 2$  super-Yang-Mills theory [1] and the *Maldacena conjecture* stating that  $\mathcal{N} = 4$  super-conformal  $SU(N_c)$ -gauge theories arising on parallel D3-branes are in the limit of large 't Hooft coupling and large  $N_c$  dual to supergravity theories on an  $AdS_5$ -background [2]. Despite of these striking results there is still a long way to go towards a better understanding of nonperturbative effects in supersymmetric theories with less supersymmetries and finite  $N_c$ . In particular, since low-energy physics is manifestly not supersymmetric, it is necessary that this symmetry is broken at some energy scale. As issues of supersymmetry breaking are difficult to address in perturbation theory, one is motivated to study supersymmetric models on a *spacetime lattice*. Unfortunately, supersymmetry is explicitly broken by most discretization procedures, and it is a nontrivial problem to recover supersymmetry in the continuum limit. However, there are discretizations with nonlocal interaction terms for which supersymmetry is manifestly realized [3]. Alternatively, for some models one can discretize space – but not time – such that a subalgebra of the supersymmetry algebra which determines spectral properties of the super-Hamiltonian remains intact [4].

Every supersymmetric field theory on a spatial lattice may be reinterpreted as a higher-dimensional *supersymmetric quantum mechanical system*. The first studies of such systems go back to Nicolai [5] and have been extended by Witten in his work on supersymmetry breaking [6, 7, 8]. Soon after that, de Crombrugghe and Rittenberg [9] presented a very general analysis of supersymmetric Hamiltonians. Over the years, it has been demonstrated that supersymmetry is a useful technique to construct exact solutions in quantum mechanics [10]. For example, all ordinary Schrödinger equations with shape invariant potentials can be solved algebraically with the methods of supersymmetry. On the other hand, apparently different quantum systems may be related by supersymmetry, and this relation may shed new light on the physics of the two systems. For example, the hydrogen atom (its Hamiltonian, angular momentum and Runge-Lenz vector) can be supersymmetrized. The corresponding theory contains both the proton-electron and the proton-positron system as subsectors [11].

The present work contains the first part of our attempt to better understand supersymmetric field theories on spatial lattices. Here, we will analyze properties of quantum mechanical systems. In a forthcoming publication, our results will be related to Wess-Zumino models on such lattices. This paper is organized as follows: In Section 2 we recall supersymmetric quantum mechanics with  $\mathcal{N}$  supersymmetries. The main emphasis is on the algebraic structure of such systems. In the following section we give explicit realizations of systems with one, two or more supersymmetries. They are based on the Dirac operator in external gauge and gravitational fields. We shall see that for certain background fields there are  $\mathcal{N}$  inequivalent ways to take the square root of  $-\mathcal{D}^2$ . At the

same time  $-D^2$  commutes with several particle-number operators which correspond to complex structures. The superalgebra implies consistency conditions for these structures and the gauge field strength. For example, the Dirac operator in four dimensions admits an extended  $\mathcal{N} = 4$  supersymmetry if spacetime is hyper-Kähler and the gauge field is (anti-)selfdual. In Section 4 we show that, for background fields admitting an extended supersymmetry, the geometry and gauge potential are encoded in a superpotential. The superpotential may be used to deform the generally- and gauge-covariant derivative into the ordinary derivative. In Section 5 we apply our general results to study the Dirac operator on the complex projective spaces  $\mathbb{C}P^n$  with an Abelian background gauge field. We derive explicit expressions for the superpotential and fermionic zero modes on these Kähler spaces.

## 2 Extended Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics describes systems with nonnegative Hamiltonians that can be written as

$$\delta_{ij}H = \frac{1}{2}\{Q_i, Q_j\}, \quad i, j = 1, \dots, \mathcal{N}, \quad (1)$$

with *Hermitian* supercharges  $Q_i$  anticommuting with an involutory operator  $\Gamma$ ,

$$\{Q_i, \Gamma\} = 0, \quad \Gamma^\dagger = \Gamma, \quad \Gamma^2 = \mathbb{1}. \quad (2)$$

There are various definitions of supersymmetric quantum mechanics in the literature, for a recent discussion, in particular concerning the role of the grading operator  $\Gamma$ , we refer to [12]. One may also relax the condition for the left-hand side of (1), see for example [13], but in this paper we will not consider such systems.

The  $+1$  and  $-1$  eigenspaces of  $\Gamma$  are called *bosonic* and *fermionic sectors* respectively,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F, \quad \mathcal{H}_B = \mathcal{P}_+\mathcal{H}, \quad \mathcal{H}_F = \mathcal{P}_-\mathcal{H}, \quad \mathcal{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma). \quad (3)$$

The supercharges  $Q_i$  map  $\mathcal{H}_B$  into  $\mathcal{H}_F$  and vice versa. The super-algebra (1) implies that they commute with the super-Hamiltonian,

$$[Q_i, H] = 0, \quad (4)$$

and generate supersymmetries of the system. The simplest models exhibiting this structure are  $2 \times 2$ -matrix Schrödinger operators in one dimension [5, 6, 7]. In this paper we shall investigate explicit representations of the superalgebra (1) with one, two, four and more supercharges.

**One supercharge:** In this case every eigenstate of  $H = Q_1^2 \geq 0$  with positive energy is paired by the action of  $Q_1$ . For example, if  $|B\rangle$  is a bosonic eigenstate with positive energy, then  $|F\rangle \sim Q_1|B\rangle$  is a fermionic eigenstate with the same energy. However, a normalizable eigenstate with zero energy is annihilated by the supercharge,  $Q_1|0\rangle = 0$ , and hence has no superpartner. In a basis where  $\Gamma = \sigma_3 \otimes \mathbb{1}$ , the Hermitian charge  $Q_1$  has the form

$$Q_1 = \mathcal{P}_- Q_1 \mathcal{P}_+ + \mathcal{P}_+ Q_1 \mathcal{P}_- \equiv \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}. \quad (5)$$

The index of  $Q_1$  counts the difference of bosonic and fermionic zero modes,

$$\text{ind } Q_1 = \dim \ker A - \dim \ker A^\dagger = n_B^0 - n_F^0. \quad (6)$$

Supersymmetry is spontaneously broken if and only if there exists no state which is left invariant by the supercharges, or equivalently if 0 is not in the discrete spectrum of  $H$ .

**Two supercharges:** In this case there exist two anticommuting and Hermitian roots of the super-Hamiltonian

$$H = Q_1^2 = Q_2^2, \quad \{Q_1, Q_2\} = 0, \quad Q_i^\dagger = Q_i. \quad (7)$$

Later we shall use the nilpotent *complex supercharge*

$$Q = \frac{1}{2}(Q_1 + iQ_2), \quad (8)$$

and its adjoint  $Q^\dagger$ , in terms of which the supersymmetry algebra takes the form

$$H = \{Q, Q^\dagger\}, \quad Q^2 = Q^{\dagger 2} = 0 \quad \text{and} \quad [Q, H] = 0. \quad (9)$$

The number of normalizable zero modes of  $H$  is given by [6]

$$n^0 = n_B^0 + n_F^0 = \dim(\ker Q / \text{im } Q) = \dim(\ker Q^\dagger / \text{im } Q^\dagger). \quad (10)$$

**Four supercharges:** Now there are four distinct roots of the super-Hamiltonian,

$$H = Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2. \quad (11)$$

The only nontrivial anticommutators of the complex nilpotent supercharges

$$Q = \frac{1}{2}(Q_1 + iQ_2) \quad \text{and} \quad \tilde{Q} = \frac{1}{2}(Q_3 + iQ_4) \quad (12)$$

and their adjoints are

$$\{Q, Q^\dagger\} = \{\tilde{Q}, \tilde{Q}^\dagger\} = H. \quad (13)$$

### 3 Supersymmetries and the Euclidean Dirac Operator

There exists a fundamental supersymmetric Hamiltonian in nature, the square of the Euclidean Dirac operator. In other words, one identifies the Dirac operator as a supercharge associated to this Hamiltonian. There are non-linear sigma-models which give rise to exactly these supercharges, in particular the  $(1+0)$ -dimensional models studied in [13, 14]. In contrast to those models, we allow for the presence of gauge fields but do not include torsion. The identification of Dirac operators and supercharges has also been employed by Alvarez-Gaumé in [15], where he uses supersymmetry to derive the Atiyah-Singer index theorem.

The *chiral supersymmetry* with one charge exists in all even dimensions and for arbitrary gauge and gravitational background fields. It can be extended if  $\not{D}^2$  commutes with certain *particle-number operators* to be defined below. For example, in  $D = 2n$  Euclidean dimensions and for background fields with *holonomy group*  $U(n)$  the operator  $\not{D}^2$  commutes with one particle-number operator and admits two supersymmetries. In  $D = 4n$  dimensions and for background fields with *holonomy group*  $Sp(n)$ , there are three conserved number operators and four supersymmetries.

We consider a smooth Riemannian manifold  $\mathcal{M}$  of dimension  $D$  which allows for a spin structure. We describe the gravitational fields in terms of vielbeins  $E_M^A$  rather than a metric  $G_{MN}$ , which is related to the vielbein by

$$G_{MN} = E_M^A E_N^B \delta_{AB}, \quad \delta^{AB} = G^{MN} E_M^A E_N^B. \quad (14)$$

The Lorentz indices  $A, B \in \{1, \dots, D\}$  are converted into coordinate indices  $M, N \in \{1, \dots, D\}$  (or vice versa) with the help of the vielbein  $E_M^A$  or its inverse, which is given by  $E_A^M = G^{MN} E_N^B \delta_{BA}$ . The Clifford algebra is generated by the Hermitian matrices  $\Gamma^A$ , satisfying

$$\{\Gamma^A, \Gamma^B\} = 2\delta^{AB} \quad \text{or} \quad \{\Gamma^M, \Gamma^N\} = 2G^{MN}, \quad (15)$$

where the  $\Gamma^M = \Gamma^A E_A^M$  are the matrices with respect to the holonomic basis  $\partial_M$ .

#### 3.1 Chiral Supersymmetry

In even dimensions we always have *chiral supersymmetry* generated by the Hermitian Euclidean Dirac operator, viewed as supercharge

$$Q_1 = i\not{D} = i\Gamma^M \nabla_M = i\Gamma^A \nabla_A, \quad \nabla_A = E_A^M \nabla_M. \quad (16)$$

The generally- and gauge-covariant derivative acting on spinors,

$$\begin{aligned} \nabla_M &= \partial_M + \Omega_M + A_M \\ &= \partial_M + \frac{1}{4} \Omega_{MAB} \Gamma^{AB} + A_M^a T_a, \end{aligned} \quad (17)$$

contains the connection  $\Omega$  and gauge potential  $A$  together with the anti-Hermitian generators  $\Gamma^{AB} = \frac{1}{2} [\Gamma^A, \Gamma^B]$  and  $T^a$  of spin rotations and gauge transformations. The gamma-matrices are covariantly constant in the following sense,

$$\nabla_M \Gamma^N = \partial_M \Gamma^N + \Gamma_{MP}^N \Gamma^P + [\Omega_M, \Gamma^N] = 0. \quad (18)$$

For the involutory operator  $\Gamma$  in (2) we take in  $D=2n$  dimensions

$$\Gamma = \alpha \Gamma^1 \dots \Gamma^D, \quad (19)$$

where the phase  $\alpha$  is chosen such that  $\Gamma$  is Hermitian and squares to  $\mathbb{1}$ ,  $\alpha^2 = (-)^n$ . The ‘bosonic’ and ‘fermionic’ subspaces consist of spinor fields with positive and negative chiralities, respectively, and the number of bosonic minus the number of fermionic zero modes equals the index of the Dirac operator,

$$n_B^0 - n_F^0 = \text{ind } i\mathcal{D}. \quad (20)$$

Since the commutator of two covariant derivatives yields the gauge field strength and curvature tensor in the spinor-representation,

$$\begin{aligned} [\nabla_M, \nabla_N] &= \mathcal{F}_{MN} = F_{MN} + R_{MN}, \\ F_{MN} &= \partial_M A_N - \partial_N A_M + [A_M, A_N] = F_{MN}^a T_a, \\ R_{MN} &= \partial_M \Omega_N - \partial_N \Omega_M + [\Omega_M, \Omega_N] = \frac{1}{4} R_{MNAB} \Gamma^{AB}, \end{aligned} \quad (21)$$

where the Riemann curvature tensor is obtained from the connection via

$$R_{MNAB} = \partial_M \Omega_{NAB} - \partial_N \Omega_{MAB} + \Omega_{MA}^C \Omega_{NCB} - \Omega_{NA}^C \Omega_{MCB}, \quad (22)$$

we find the squared Dirac operator or super-Hamiltonian

$$-\mathcal{D}^2 = H = Q_1^2 = -G^{MN} \nabla_M \nabla_N - \frac{1}{2} \Gamma^{AB} \mathcal{F}_{AB}. \quad (23)$$

Here we have used the components of  $\mathcal{F}_{MN}$  with respect to an orthonormal vielbein,

$$\mathcal{F}_{AB} = E_A^M E_B^N \mathcal{F}_{MN} = [\nabla_A, \nabla_B].$$

Note that the two covariant derivatives  $\nabla_M \nabla_N$  in (23) act on different types of fields. The derivative on the right acts on spinors and is given in (17), whereas the derivative on the left acts on spinors with a coordinate index and hence contains an additional term proportional to the Christoffel symbols,

$$\nabla_M \psi_N = \partial_M \psi_N + \Omega_M \psi_N - \Gamma_{MN}^P \psi_P + A_M \psi_N. \quad (24)$$

### 3.2 Extended Supersymmetries

In this section we show that for particular background fields the chiral supersymmetry can be extended to finer, particle-number conserving supersymmetries. The existence of a single conserved number operator is equivalent to the existence of a covariantly conserved complex structure. This way one finds that  $\mathcal{N}=2$  is only possible if space admits a Kähler structure, and  $\mathcal{N}=4$ , if it admits a hyper-Kähler structure. Analogous conditions are derived for the background gauge field.

### 3.2.1 Square Roots of $H = -\not{D}^2$

In this subsection we characterize a class of first order differential operators which square to  $H = -\not{D}^2$ . Our ansatz is motivated by previous results in [16, 17, 18, 19] and the simple observation that both the free Dirac operator  $\not{D}$  on flat space and

$$I_N^M \Gamma^N \partial_M$$

have the same square for any orthogonal matrix  $I$ . Thus, we are led to the following ansatz for the supercharge in a gravitational and gauge field background,

$$Q(I) = i I_N^M \Gamma^N \nabla_M \equiv i (I\Gamma)^M \nabla_M, \quad (25)$$

where  $I$  is a real tensor field with components  $I_N^M$ .

This construction is close in spirit to the one presented in [9]. The algebraic approach there is applied to the particular situation of a Dirac operator, and this will allow us to interpret all quantities in [9] as geometrical ones, like connections, curvature etc.

To derive the conditions on  $I$  and the background such that  $Q(I)^2 = H$ , we first consider the anticommutator of two operators with different  $I$ ,

$$\begin{aligned} \{Q(I), Q(J)\} &= -\frac{1}{2}(IJ^T + JI^T)^{MN} \{\nabla_M, \nabla_N\} - \frac{1}{2}\Gamma^{MN} (I^T \mathcal{F}J + J^T \mathcal{F}I)_{MN} \\ &\quad - \{(I\Gamma)^P \nabla_P (J\Gamma)^Q + (J\Gamma)^P \nabla_P (I\Gamma)^Q\} \nabla_Q, \end{aligned} \quad (26)$$

where, for example

$$(IJ^T)^{MN} = I_P^M J^{NP} \quad \text{and} \quad (I^T \mathcal{F}J)_{MN} = I_P^M \mathcal{F}_{PQ} J_N^Q.$$

After setting  $I = J$  we see that  $Q(I)$  squares to the Hamiltonian  $H$  in (23) if and only if the following three conditions are satisfied,

$$G^{MN} = (II^T)^{MN}, \quad (27)$$

$$\mathcal{F}_{MN} = (I^T \mathcal{F}I)_{MN}, \quad (28)$$

$$0 = \nabla_M I^P_Q. \quad (29)$$

By interpreting the  $I_N^M$  as components of a map  $I$  between sections of the tangent bundle, the condition (27) just means that  $I$  is an isometry,

$$G(IX, IY) = G(X, Y).$$

In view of our remarks above it should not be surprising that the components  $I^A_B$  with respect to an orthonormal vielbein form an orthogonal matrix.

The condition (29) means, that the tensor field  $I$  must be covariantly constant. With  $R_{MNAB} = R_{ABMN} = -R_{NMAB}$  the corresponding integrability conditions read

$$\begin{aligned} 0 &= I_M^R [\nabla_A, \nabla_B] I_{RN} = I_M^R R_{RSAB} I_N^S - R_{MNAB} \\ \text{or} \quad R_{MN} &= (I^T R I)_{MN}, \end{aligned}$$

and (28) simplifies to the same condition with  $\mathcal{F}_{MN}$  replaced by the gauge field strength  $F_{MN}$ . Thus we end up with the following

**Lemma:** *The charge*

$$Q(I) = i I_N^M \Gamma^N \nabla_M, \quad I_N^M(x) \in \mathbb{R}, \quad (30)$$

*is Hermitian and squares to  $H$  in (23) if and only if the following conditions hold:*

$$\nabla I = 0, \quad I I^T = \mathbb{1} \quad \text{and} \quad [I, F] = 0. \quad (31)$$

The hermiticity follows from  $\nabla I = 0$ , which in turn implies that the  $I_N^M \Gamma^N$  commute with the covariant derivative. Because of the second condition in (31) the last one is equivalent to (28) with  $\mathcal{F}$  replaced by  $F$ .

A trivial solution is of course  $I = \mathbb{1}$  in which case  $Q$  becomes the Dirac operator itself. Let us now assume that there is a second square root  $Q(I)$  anticommuting with the Dirac operator  $Q(\mathbb{1})$ . With  $\nabla I = 0$  and (26) these two charges anticommute if

$$\{Q(\mathbb{1}), Q(I)\} = -\frac{1}{2}(I^{MN} + I^{NM})\{\nabla_M, \nabla_N\} - \frac{1}{2}\Gamma^{MN} (I^T \mathcal{F} + \mathcal{F} I)_{MN} \stackrel{!}{=} 0,$$

and this shows that the map  $I$  must be antisymmetric. Because of (27) it squares to  $-\mathbb{1}$ . Hence it defines an *almost complex structure*. Since it is covariantly constant, the manifold is Kähler with complex structure  $I$ . Thus we have shown that  $H$  admits two supersymmetries generated by  $Q(\mathbb{1})$  and  $Q(I)$  if the manifold is Kähler with complex structure  $I$  and if the gauge field strength  $F$  commutes with this structure.

Now we are ready to generalize to  $\mathcal{N}$  supercharges

$$Q(\mathbb{1}) \quad \text{and} \quad Q(I_i), \quad i = 1, \dots, \mathcal{N} - 1.$$

From our general result (26) we conclude

**Lemma:** *The  $\mathcal{N}$  charges*

$$Q(\mathbb{1}) = i \not{D} \quad \text{and} \quad Q(I_i) = i I_i^M{}_N \Gamma^N \nabla_M, \quad i = 1, \dots, \mathcal{N} - 1 \quad (32)$$

*are Hermitian and generate an extended superalgebra (1), if and only if*

$$\{I_i, I_j\} = -2\delta_{ij} \mathbb{1}_D, \quad I_i^T = -I_i, \quad (33)$$

$$\nabla I_i = 0, \quad [I_i, F] = 0. \quad (34)$$

The covariantly conserved complex structures form a  $D$ -dimensional real representation of the Euclidean Clifford algebra with  $\mathcal{N} - 1$  gamma-matrices. We call a representation



irreducible, if only  $\mathbb{1}$  commutes with all those matrices. Irreducible representations are known to exist for

$$\begin{aligned}\mathcal{N} - 1 &= 8n, & 6 + 8n, & 7 + 8n, \\ D &= 16^n, & 8 \cdot 16^n, & 8 \cdot 16^n,\end{aligned}\tag{35}$$

with  $n \in \mathbb{N}_0$ . In these cases, only trivial gauge fields on flat space are possible. We further observe that, if  $\{I_1, \dots, I_k, F\}$  satisfy the conditions in the above lemma, then also  $\{I_1, \dots, I_k, I_{k+1} = I_1 \cdots I_k, F\}$  do, provided

$$k = 2 + 4n.$$

It follows, for example, that the superalgebra with 7 supercharges  $\{Q(\mathbb{1}), Q(I_i)\}$  can always be extended to a superalgebra with 8 supercharges. In addition, since for  $\mathcal{N} - 1 = 8n$  the Euclidean gamma-matrices may be chosen real and chiral, one can construct the corresponding complex structures out of the complex structures  $\tilde{I}_i$  of the  $\mathcal{N} = 8n$  supersymmetry,

$$I_i = \begin{pmatrix} 0 & \tilde{I}_i \\ \tilde{I}_i & 0 \end{pmatrix} \quad \text{and} \quad I_{8n} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

For irreducible  $I_i$  one can construct systems with

$$\mathcal{N} = 8n \quad \text{and} \quad \mathcal{N} = 8n + 1\tag{36}$$

independent real supercharges in this way. Note, however, that there may exist  $D$ -dimensional matrices  $I_i$  which do not generate all of  $\text{GL}(D)$  and hence do not belong to a real irreducible representation of the Clifford algebra. These are the most interesting cases since they admit nontrivial background fields commuting with all  $I_i$ , as required in our lemma above. Below we will discuss such systems with  $\mathcal{N} = 4 + 8n$ .

### 3.2.2 $\mathcal{N}=2$ and Particle-Number Operator

On any Kähler manifold the Dirac operator admits an extended  $\mathcal{N}=2$  supersymmetry if the field strength commutes with the complex structure. With respect to a suitably chosen orthonormal frame the structure has the form  $(I^A_B) = i\sigma_2 \otimes \mathbb{1}_n$ . The charge  $Q(I)$  on a Kähler manifold with complex structure  $I$  squares to  $H$  and commutes with the Dirac operator if and only if

$$[I, F] = 0 \quad \text{or} \quad (F_{AB}) = \begin{pmatrix} U & V \\ -V & U \end{pmatrix}, \quad U^T = -U, \quad V^T = V.$$

The complex nilpotent charge in (8) takes the simple form

$$Q = \frac{1}{2}Q(\mathbb{1}) + \frac{i}{2}Q(I) = i\psi^A \nabla_A\tag{37}$$

with operators

$$\psi^A = P^A_B \Gamma^B, \quad P^A_B = \frac{1}{2}(\mathbb{1} + iI)^A_B. \quad (38)$$

$P$  projects onto the  $n$ -dimensional  $I$ -eigenspace corresponding to the eigenvalue  $-i$ , its complex conjugate  $\bar{P}$  onto the  $n$ -dimensional eigenspace  $+i$ . The two eigenspaces are complementary and orthogonal,  $P + \bar{P} = \mathbb{1}$  and  $P\bar{P} = 0$ . The  $\psi^A$  and their adjoints form a fermionic algebra,

$$\{\psi^A, \psi^B\} = \{\psi^{A\dagger}, \psi^{B\dagger}\} = 0 \quad \text{and} \quad \{\psi^A, \psi^{B\dagger}\} = 2P^{AB}. \quad (39)$$

At this point it is natural to introduce the *number operator*

$$N = \frac{1}{2}\psi_A^\dagger \psi^A = \frac{1}{4}(D + iI_{AB}\Gamma^{AB}), \quad (40)$$

whose eigenvalues are lowered and raised by  $\psi^A$  and  $\psi^{A\dagger}$ , respectively,

$$\begin{aligned} [N, \psi^{A\dagger}] &= P^A_B \psi^{B\dagger} = \psi^{A\dagger}, \\ [N, \psi^A] &= -P^A_B \psi^B = -\psi^A. \end{aligned} \quad (41)$$

Since only  $n = \text{rank } P$  of the  $2n$  creation operators are linearly independent we have inserted a factor  $\frac{1}{2}$  in the definition of the number operator  $N$  in (40). This operator commutes with the covariant derivative, because  $\nabla I = 0$  is equivalent to

$$[\nabla_M, N] = \partial_M N + [\Omega_M, N] = 0, \quad (42)$$

and therefore  $Q$  decreases  $N$  by one, while its adjoint  $Q^\dagger$  increases it by one,

$$[N, Q] = -Q \quad \text{and} \quad [N, Q^\dagger] = Q^\dagger. \quad (43)$$

The corresponding real supercharges are given by

$$\begin{aligned} Q(\mathbb{1}) &= Q + Q^\dagger = i\not{D}, \\ Q(I) &= i(Q^\dagger - Q) = i[N, i\not{D}]. \end{aligned} \quad (44)$$

Finally, we observe that the Hermitian matrix

$$\Sigma = N - \frac{1}{4}D = \frac{i}{4}I_{AB}\Gamma^{AB} \in \text{spin}(D)$$

generates a  $U(1)$  subgroup of  $\text{Spin}(D)$ . This is the R-symmetry of the superalgebra,

$$\begin{pmatrix} Q(\mathbb{1}) \\ Q(I) \end{pmatrix} \longrightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} Q(\mathbb{1}) \\ Q(I) \end{pmatrix}.$$

Next, we introduce the *Clifford vacuum*  $|0\rangle$ , which is annihilated by all annihilation operators  $\psi^A$  and hence has particle number  $N = 0$ . The corresponding Clifford space

$\mathcal{C}$  is the Fock space built over this vacuum state. Since only  $n$  creation operators are linearly independent, we obtain the following grading of the Clifford space,

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \dots \oplus \mathcal{C}_n, \quad \dim \mathcal{C}_p = \binom{n}{p}, \quad (45)$$

with subspaces labelled by their particle number,

$$N|_{\mathcal{C}_p} = p \cdot \mathbb{1}. \quad (46)$$

In particular, the one-dimensional subspace  $\mathcal{C}_0$  is spanned by  $|0\rangle$  and the  $n$ -dimensional subspace  $\mathcal{C}_1$  by the linearly dependent states  $\psi_A^\dagger |0\rangle$ . Along with the Clifford space the Hilbert space of all square integrable spinor fields decomposes as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n \quad \text{with} \quad N|_{\mathcal{H}_p} = p \cdot \mathbb{1}. \quad (47)$$

Similar to the standard Fock space construction, cf. e.g. [20], the number operator  $N$  in (40) commutes with the Hamiltonian  $H$  even in curved space and in the presence of gauge fields. Thus,  $H$  leaves  $\mathcal{H}_p$  invariant. The nilpotent charge  $Q$  maps  $\mathcal{H}_p$  into  $\mathcal{H}_{p-1}$  and its adjoint  $Q^\dagger$  maps  $\mathcal{H}_p$  into  $\mathcal{H}_{p+1}$ .

The raising and lowering operators  $\psi^{A\dagger}$  and  $\psi^A$  are linear combinations of  $\Gamma^A$  and therefore anticommute with  $\Gamma$  in (19). Hence they map left- into right-handed spinors and vice versa. Since  $\Gamma|0\rangle$  is annihilated by all  $\psi^A$ ,

$$\psi^A(\Gamma|0\rangle) = -\Gamma\psi^A|0\rangle = 0,$$

and since the Clifford vacuum  $|0\rangle$  is unique, we conclude that  $|0\rangle$  has definite chirality. It follows that all states with even  $N$  have the same chirality as  $|0\rangle$ , and all states with odd  $N$  have opposite chirality,

$$\Gamma = \pm(-)^N. \quad (48)$$

### 3.2.3 $\mathcal{N} = 3$ and $\mathcal{N} = 4$ Superalgebras

If  $\{I_1, I_2\}$  satisfy the conditions (33,34), then  $\{I_1, I_2, I_3 = \pm I_1 I_2\}$  do so as well. For this reason  $\mathcal{N} = 3$  supersymmetry implies automatically  $\mathcal{N} = 4$  supersymmetry. Hence, it suffices to consider systems with 4 supercharges. This should be compared to the results in [13], where systems with  $\mathcal{N} = 3$  but  $\mathcal{N} \neq 4$  are possible, the reason for this being that in [13] a more general algebra than (1) has been studied.

The dimension of the matrices  $I_i$  (which equals the dimension of the manifold) must be a multiple of 4,  $D = 4n$ . In this section we choose the selfdual or anti-selfdual matrices,

$$\begin{aligned} \text{SD:} \quad & \tilde{I}_1 = i\sigma_0 \otimes \sigma_2, & \tilde{I}_2 = i\sigma_2 \otimes \sigma_3, & \tilde{I}_3 = i\sigma_2 \otimes \sigma_1 = -\tilde{I}_1 \tilde{I}_2, \\ \text{ASD:} \quad & \tilde{I}_1 = i\sigma_3 \otimes \sigma_2, & \tilde{I}_2 = i\sigma_2 \otimes \sigma_0, & \tilde{I}_3 = i\sigma_1 \otimes \sigma_2 = \tilde{I}_1 \tilde{I}_2, \end{aligned} \quad (49)$$

and define  $I_i = \tilde{I}_i \otimes \mathbb{1}_n$ . They generate two commuting  $so(3)$  subalgebras of  $so(4n)$ . The conditions (33,34) imply that the curvature tensor ( $R_{AB}$ ) and gauge field strength ( $F_{AB}$ ) commute with all three  $I_i$ . For example, in 4 dimensions both must be *selfdual* or *anti-selfdual*. A 4-dimensional manifold with (anti-)selfdual curvature is hyper-Kähler. More generally, a  $4n$ -dimensional manifold is hyper-Kähler if it admits three covariantly constant and anticommuting complex structures. We see, that  $-\not{D}^2$  admits 4 supersymmetries if and only if the underlying space  $\mathcal{M}$  is hyper-Kähler and the gauge field strength commutes with the three complex structures.

For each complex structure  $I_i$  there exists an associated number operator

$$N_i = N(I_i) = \frac{1}{4}D + \Sigma(I_i), \quad \Sigma(I) = \frac{i}{4}I_{AB}\Gamma^{AB}, \quad (50)$$

and the 4 real supercharges take the form

$$Q(\mathbb{1}) = i\not{D} \quad \text{and} \quad Q(I_i) = i[N_i, i\not{D}]. \quad (51)$$

However, the 3 number operators do not commute, because

$$[\Sigma(I_i), \Sigma(I_j)] = i\Sigma([I_i, I_j]), \quad (52)$$

and the antisymmetric matrices  $I_i$ , together with  $\mathbb{1}_{4n}$ , form a  $4n$ -dimensional real representation of the *non-commutative* quaternionic algebra,

$$I_i I_j = -\delta_{ij} \mathbb{1}_{4n} \pm \epsilon_{ijk} I_k. \quad (53)$$

The three matrices  $\Sigma(I_i)$  generate an  $SO(3)$ -subgroup of  $Spin(4n)$  which rotates the real supercharges. This is proven with the help of the simple identities

$$i[\Sigma(I), Q(\mathbb{1})] = Q(I) \quad \text{and} \quad i[\Sigma(I), Q(J)] = Q(JI).$$

Now it follows at once, that the selfdual (anti-selfdual)  $SO(3)$ -subgroup of the  $SO(4)$  R-symmetry is implemented by the exponentiated action of the  $\Sigma(I_i)$ ,

$$U(\vec{\alpha})Q_m U^{-1}(\vec{\alpha}) = R_{mn}Q_n, \quad \text{where} \\ U(\vec{\alpha}) = \exp(i\Sigma(\alpha_i I_i)), \quad R(\vec{\alpha}) = \exp(\alpha_i \tilde{I}_i).$$

The  $\tilde{I}_i$  are the 4-dimensional selfdual (anti-selfdual) matrices in (49), and  $I_i = \tilde{I}_i \otimes \mathbb{1}_n$  are  $4n$ -dimensional complex structures with respect to a suitable orthonormal base. The  $Q_m$  are the four real supercharges,

$$\{Q_0, Q_1, Q_2, Q_3\} \equiv \{Q(\mathbb{1}), Q(I_1), Q(I_2), Q(I_3)\}.$$

Let us remark, that other choices for the complex structures than those obtained from (49) are possible.

### 3.2.4 $\mathcal{N} = 7$ and $\mathcal{N} = 8$ Superalgebras

According to (35) we can find 6 or 7 real and antisymmetric matrices  $I_i$ , for example the 8-dimensional (irreducible) matrices

$$\begin{aligned}\tilde{I}_1 &= i\sigma_1 \otimes \sigma_0 \otimes \sigma_2, & \tilde{I}_3 &= i\sigma_2 \otimes \sigma_1 \otimes \sigma_0, & \tilde{I}_5 &= i\sigma_0 \otimes \sigma_2 \otimes \sigma_1, \\ \tilde{I}_2 &= i\sigma_3 \otimes \sigma_0 \otimes \sigma_2, & \tilde{I}_4 &= i\sigma_2 \otimes \sigma_3 \otimes \sigma_0, & \tilde{I}_6 &= i\sigma_0 \otimes \sigma_2 \otimes \sigma_3, \\ \tilde{I}_7 &= \tilde{I}_1 \tilde{I}_2 \tilde{I}_3 \tilde{I}_4 \tilde{I}_5 \tilde{I}_6 = -i\sigma_2 \otimes \sigma_2 \otimes \sigma_2,\end{aligned}\tag{54}$$

tensored with  $\mathbb{1}_n$ . Thus we can satisfy (33) in  $8n$  dimensions and a  $\mathcal{N} = 7$  superalgebra can always be extended to a  $\mathcal{N} = 8$  superalgebra, since if  $\{I_1, \dots, I_6, F\}$  satisfy the conditions (33,34) then  $\{I_1, \dots, I_6, I_7 = I_1 \cdots I_6, F\}$  do so as well.

In 8 dimensions there is no non-trivial solution to

$$[I_i, \mathcal{F}] = 0, \quad i = 1, \dots, 7,$$

since the only matrix commuting with all  $I_i$  in (54) is the identity matrix. Hence the manifold must be flat and the gauge field strength must vanish. In 8 dimensions, only the free Dirac operator admits an  $\mathcal{N} = 8$  supersymmetry. However, in  $8n$  dimensions with  $n = 2, 3, \dots$ , there are nontrivial solutions to the constraints in (33,34). For example, every field strength  $(F_{AB}) = \mathbb{1}_8 \otimes \tilde{F}$  with antisymmetric  $\tilde{F}$  commutes with the  $I_i$  listed in (54). In the case of extended supersymmetry one can define a set  $\{N_{01}, N_{23}, N_{45}, \dots\}$  of particle-number operators that commute with each other and with the Hamiltonian. Here the  $N_{ij}$  are defined as

$$N_{ij} = N(I_i I_j) = \frac{1}{4}D + \Sigma(I_i I_j), \quad \text{where } I_0 = \mathbb{1}.\tag{55}$$

## 4 Superpotentials on Kähler Manifolds

The super-Hamiltonian  $-\mathcal{D}^2$  admits an extended supersymmetry if it commutes with a number operator or if the complex supercharge is nilpotent and decreases the particle number by one. Then the manifold is Kähler and the complex structure commutes with the gauge field strength. Now we shall see that this in turn is the condition for the existence of a superpotential  $g$  from which the spin connection and gauge potential can be derived.

Kähler manifolds of real dimension  $D = 2n$  are particular complex manifolds and we may introduce complex coordinates  $(z^\mu, \bar{z}^{\bar{\mu}})$  with  $\mu, \bar{\mu} = 1, \dots, n$  [21]. The real and complex coordinate differentials are related as follows

$$\begin{aligned}dz^\mu &= \frac{\partial z^\mu}{\partial x^M} dx^M \equiv f^\mu_M dx^M, & d\bar{z}^{\bar{\mu}} &= \frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^M} dx^M \equiv f^{\bar{\mu}}_M dx^M, \\ \partial_\mu &= \frac{\partial x^M}{\partial z^\mu} \partial_M \equiv f^M_\mu \partial_M, & \partial_{\bar{\mu}} &= \frac{\partial x^M}{\partial \bar{z}^{\bar{\mu}}} \partial_M \equiv f^M_{\bar{\mu}} \partial_M.\end{aligned}\tag{56}$$

The integrability conditions for the  $dz^\mu$  to be differentials of complex coordinate functions  $z^\mu$  are automatically satisfied on a Kähler manifold.

The  $f^\mu$  and  $f_\mu$  are left and right eigenvectors of the complex structure,

$$f^\mu_M I^M_N = -i f^\mu_N \quad \text{and} \quad I^M_N f^N_\mu = -i f^M_\mu, \quad \mu = 1, \dots, n. \quad (57)$$

Since  $I^M_N$  is antisymmetric with respect to the scalar product  $(A, B) = A_M G^{MN} B_N$ , the eigenvectors with different eigenvalues are orthogonal in the following sense,

$$G^{MN} f^\mu_M f^\nu_N = G_{MN} f^M_\mu f^N_\nu = 0. \quad (58)$$

Identity and complex structure possess the spectral decompositions

$$\delta^M_N = f^M_\mu f^\mu_N + f^M_{\bar{\mu}} f^{\bar{\mu}}_N, \quad (59)$$

$$iI^M_N = f^M_\mu f^\mu_N - f^M_{\bar{\mu}} f^{\bar{\mu}}_N, \quad (60)$$

and the relations  $\partial z^\mu / \partial z^\nu = \delta^\mu_\nu$  and  $\partial z^\mu / \partial \bar{z}^{\bar{\nu}} = 0$  translate into

$$f^\mu_M f^M_\nu = \delta^\mu_\nu \quad \text{and} \quad f^\mu_M f^M_{\bar{\nu}} = 0. \quad (61)$$

With (58) the line element takes the form

$$ds^2 = G_{MN} dx^M dx^N = 2h_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}}, \quad h_{\mu\bar{\nu}} = h_{\bar{\nu}\mu} = G_{MN} f^M_\mu f^N_{\bar{\nu}}, \quad (62)$$

where the  $h_{\mu\bar{\nu}}$  are derived from a real Kähler potential  $K$  as follows,

$$h_{\mu\bar{\nu}} = \frac{\partial^2 K}{\partial z^\mu \partial \bar{z}^{\bar{\nu}}} \equiv \partial_\mu \partial_{\bar{\nu}} K. \quad (63)$$

Covariant and exterior derivatives split into holomorphic and antiholomorphic pieces,

$$\begin{aligned} \nabla &= dz^\mu \nabla_\mu + d\bar{z}^{\bar{\mu}} \nabla_{\bar{\mu}}, \\ d &= dz^\mu \partial_\mu + d\bar{z}^{\bar{\mu}} \partial_{\bar{\mu}} = \partial + \bar{\partial}, \end{aligned} \quad (64)$$

and the only non-vanishing components of the Christoffel symbols are

$$\Gamma^\rho_{\mu\nu} = h^{\rho\bar{\sigma}} \partial_\mu h_{\bar{\sigma}\nu} = h^{\rho\bar{\sigma}} \partial_{\bar{\sigma}\mu\nu} K, \quad (65)$$

$$\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} = h^{\bar{\rho}\sigma} \partial_{\bar{\mu}} h_{\sigma\bar{\nu}} = h^{\bar{\rho}\sigma} \partial_{\sigma\bar{\mu}\bar{\nu}} K. \quad (66)$$

Along with the derivatives the forms split into holomorphic and antiholomorphic parts. For example, the first Chern class  $c_1 = (2\pi i)^{-1} h_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}}$  is a  $(1, 1)$ -form and the gauge potential  $A = A_\mu dz^\mu + A_{\bar{\mu}} d\bar{z}^{\bar{\mu}}$  a sum of a  $(1, 0)$ - and a  $(0, 1)$ -form. With the help of (65) the covariant derivative of a  $(1, 0)$ -vector field can be written as

$$\begin{aligned} \nabla_\mu (B^\nu \partial_\nu) &= (\partial_\mu B^\rho + \Gamma^\rho_{\mu\nu} B^\nu) \partial_\rho \\ &= (\partial_\mu B^\rho + h^{\rho\bar{\sigma}} (\partial_\mu h_{\bar{\sigma}\nu}) B^\nu) \partial_\rho \\ &= h^{\rho\bar{\sigma}} \partial_\mu (h_{\bar{\sigma}\nu} B^\nu) \partial_\rho. \end{aligned} \quad (67)$$

Let us introduce complex vielbeins  $e_\alpha = e_\alpha^\mu \partial_\mu$  and  $e^\alpha = e_\mu^\alpha dz^\mu$ , such that  $h_{\bar{\mu}\nu} = \frac{1}{2} \delta_{\bar{\alpha}\beta} e_{\bar{\mu}}^{\bar{\alpha}} e_\nu^\beta$ . The components of the complex connection can be related to the metric  $h_{\bar{\mu}\nu}$  and the vielbeins with the help of Leibniz' rule and (65) as follows,

$$\begin{aligned}\omega_{\mu\alpha}^\beta e_\beta &\equiv \nabla_\mu e_\alpha = \nabla_\mu (e_\alpha^\nu \partial_\nu) = (\partial_\mu e_\alpha^\nu) \partial_\nu + e_\alpha^\nu \Gamma_{\mu\nu}^\rho \partial_\rho \\ &= (\partial_\mu e_\alpha^\nu) \partial_\nu + e_\alpha^\nu h^{\rho\bar{\sigma}} \partial_\mu (h_{\bar{\sigma}\nu}) \partial_\rho = h^{\rho\bar{\sigma}} \partial_\mu (e_\alpha^\nu h_{\bar{\sigma}\nu}) \partial_\rho = e_\rho^\beta h^{\rho\bar{\sigma}} \partial_\mu (e_\alpha^\nu h_{\bar{\sigma}\nu}) e_\beta.\end{aligned}$$

Comparing the coefficients of  $e_\beta$  yields the connection coefficients  $\omega_{\mu\alpha}^\beta$ . The remaining coefficients are obtained the same way, and one finds altogether

$$\begin{aligned}\omega_{\mu\alpha}^\beta &= e^{\beta\bar{\sigma}} \partial_\mu e_{\bar{\sigma}\alpha}, & \omega_{\mu\bar{\alpha}}^{\bar{\beta}} &= e_{\bar{\sigma}}^{\bar{\beta}} \partial_\mu e_{\bar{\alpha}}^{\bar{\sigma}}, \\ \omega_{\bar{\mu}\bar{\alpha}}^{\bar{\beta}} &= e_{\bar{\sigma}}^{\bar{\beta}} \partial_{\bar{\mu}} e_{\bar{\alpha}}^{\bar{\sigma}}, & \omega_{\bar{\mu}\alpha}^\beta &= e_\sigma^\beta \partial_{\bar{\mu}} e_\alpha^\sigma,\end{aligned}\tag{68}$$

where, for example,  $e^{\beta\bar{\sigma}} = h^{\bar{\sigma}\rho} e_\rho^\beta$ .

Now we are ready to rewrite the Dirac operator in complex coordinates. For that we insert the completeness relation (59) in  $i\mathcal{D} = i\Gamma^N \delta_N^M \nabla_M$  and obtain

$$i\mathcal{D} = Q + Q^\dagger \equiv 2i\psi^\mu \nabla_\mu + 2i\psi^{\dagger\bar{\mu}} \nabla_{\bar{\mu}},\tag{69}$$

where we are led to the independent fermionic raising and lowering operators,

$$\psi^\mu = \frac{1}{2} f_M^\mu \Gamma^M, \quad \psi^{\dagger\bar{\mu}} = \frac{1}{2} f_{\bar{M}}^{\bar{\mu}} \Gamma^{\bar{M}},\tag{70}$$

and the complex covariant derivatives

$$\nabla_\mu = f_M^\mu \nabla_M, \quad \nabla_{\bar{\mu}} = f_{\bar{M}}^{\bar{\mu}} \nabla_{\bar{M}}.\tag{71}$$

Of course, the supercharge  $Q$  in (69) is just the charge in (37) rewritten in complex coordinates. Contrary to the annihilation operators  $\psi^A$ , the fermionic operators  $\psi^\mu$  are independent. They fulfill the anticommutation relations

$$\{\psi^\mu, \psi^\nu\} = \{\psi^{\dagger\bar{\mu}}, \psi^{\dagger\bar{\nu}}\} \stackrel{(58)}{=} 0, \quad \{\psi^\mu, \psi^{\dagger\bar{\nu}}\} = \frac{1}{2} h^{\mu\bar{\nu}},\tag{72}$$

where  $h^{\mu\bar{\nu}} = f_M^\mu f_{\bar{M}}^{\bar{\nu}}$  is the inverse of  $h_{\mu\bar{\nu}}$  in (62). This can be seen as follows,

$$\begin{aligned}h^{\bar{\mu}\sigma} h_{\sigma\bar{\nu}} &= f_{\bar{M}}^{\bar{\mu}} f_M^\sigma \cdot f_{\bar{\sigma}}^N f_{N\bar{\nu}} \stackrel{(58)}{=} f_{\bar{M}}^{\bar{\mu}} (f_{\bar{\sigma}}^N f_M^\sigma + f_{\bar{\sigma}}^N f_{\bar{M}}^{\bar{\sigma}}) f_{N\bar{\nu}} \\ &\stackrel{(59)}{=} f_{\bar{M}}^{\bar{\mu}} f_{M\bar{\nu}} \stackrel{(61)}{=} \delta_{\bar{\nu}}^{\bar{\mu}}.\end{aligned}$$

The operators  $\psi^\mu$  lower the value of the Hermitian number operator

$$N = 2h_{\bar{\mu}\nu} \psi^{\dagger\bar{\mu}} \psi^\nu\tag{73}$$

by one, while the  $\psi^{\dagger\bar{\mu}}$  raise it by one. The proof is simple,

$$[N, \psi^\sigma] = 2h_{\bar{\mu}\nu} [\psi^{\dagger\bar{\mu}} \psi^\nu, \psi^\sigma] = -2h_{\bar{\mu}\nu} \{\psi^{\dagger\bar{\mu}}, \psi^\sigma\} \psi^\nu = -h_{\bar{\mu}\nu} h^{\bar{\mu}\sigma} \psi^\nu = -\psi^\sigma.$$

With (59,60) the fermionic operators in (38) and (70) are related as follows,

$$\begin{aligned}\psi^M &= \frac{1}{2}(\mathbb{1} + iI)^M_N \Gamma^N = 2f^M_\mu \psi^\mu, \\ \psi^{\dagger M} &= \frac{1}{2}(\mathbb{1} - iI)^M_N \Gamma^N = 2f^M_{\bar{\mu}} \psi^{\dagger \bar{\mu}},\end{aligned}\tag{74}$$

and we conclude that the number operators in (40) and (73) are indeed equal,

$$\frac{1}{2}\psi^{\dagger M}\psi_M = 2G_{MN}f^M_{\bar{\mu}}f^N_{\nu}\psi^{\dagger \bar{\mu}}\psi^\nu = 2h_{\bar{\mu}\nu}\psi^{\dagger \bar{\mu}}\psi^\nu.$$

Now we are ready to prove that in cases where  $\mathcal{D}$  admits an extended supersymmetry there exists a superpotential for the spin and gauge connections. Indeed, if spacetime is Kähler and the gauge field strength commutes with the complex structure,

$$F_{MN} = (I^T FI)_{MN},\tag{75}$$

then the complex covariant derivatives commute

$$[\nabla_\mu, \nabla_\nu] = \mathcal{F}_{\mu\nu} = f^M_\mu f^N_\nu \mathcal{F}_{MN} = 0.\tag{76}$$

One just needs to insert (60) and use (61). Alternatively, one may use  $Q^2=0$  with the complex supercharge in (69). Equation (76) is just the integrability condition (cf. Yang's equation [22]) for the existence of a superpotential  $g$  such that the complex covariant derivative can be written as

$$\nabla_\mu = g\partial_\mu g^{-1} = \partial_\mu + g(\partial_\mu g^{-1}) = \partial_\mu + \omega_\mu + A_\mu.\tag{77}$$

This useful property is true for any (possibly charged) tensor field on a Kähler manifold provided (75) holds. If the Kähler manifold admits a spin structure, as for example  $\mathbb{C}P^n$  for odd  $n$ , then (77) holds true for a (possibly charged) spinor field.

Of course, the superpotential  $g$  depends on the representation according to which the fields transform under the gauge and Lorentz group. One of the more severe technical problems in applications is to obtain  $g$  in the representation of interest. It consists of two factors,  $g = g_A g_\omega$ . The first factor  $g_A$  is the path-ordered integral of the gauge potential. According to (68) and (77) the matrix  $g_\omega$  in the vector representation is just the vielbein  $e^{\beta\bar{\sigma}}$ . If one succeeds in rewriting this  $g_\omega$  as the exponential of a matrix, then the transition to any other representation is straightforward: one contracts the matrix in the exponent with the generators in the given representation. This will be done for the complex projective spaces in the following section.

Now let us assume that we have found the superpotential  $g$ . Then we can rewrite the complex supercharge in (69) as follows,

$$Q = 2i\psi^\mu \nabla_\mu = 2igQ_0 g^{-1}, \quad Q_0 = \psi_0^\mu \partial_\mu, \quad \psi_0^\mu = g^{-1}\psi^\mu g.\tag{78}$$

The annihilation operators  $\psi^\mu$  are covariantly constant,

$$\nabla_\mu \psi^\nu = \partial_\mu \psi^\nu + \Gamma_{\mu\rho}^\nu \psi^\rho + [\omega_\mu, \psi^\nu] = 0,\tag{79}$$



and this translates into the following property of the conjugate operators,

$$\begin{aligned}\partial_\mu \psi_0^\nu &= g^{-1} (\partial_\mu \psi^\nu + [g \partial_\mu g^{-1}, \psi^\nu]) g \stackrel{(77)}{=} g^{-1} (\partial_\mu \psi^\nu + [\omega_\mu, \psi^\nu]) g \\ &= -\Gamma_{\mu\rho}^\nu g^{-1} \psi^\rho g \stackrel{(65)}{=} -h^{\nu\bar{\sigma}} (\partial_\mu h_{\bar{\sigma}\rho}) \psi_0^\rho.\end{aligned}$$

This implies the following simple equation,

$$\partial_\mu (h_{\bar{\sigma}\rho} \psi_0^\rho) = 0, \quad (80)$$

stating that the transformed annihilation operators  $\psi_{0\bar{\sigma}}$  are antiholomorphic. Indeed, one can show that they are even constant.

The relation (78) between the free supercharge  $Q_0$  and the  $g$ -dependent supercharge  $Q$  is the main result of this section. It can be used to determine zero modes of the Dirac operator. With (44) we find

$$i\not{D}\chi = 0 \quad \Longleftrightarrow \quad Q\chi = 0, \quad Q^\dagger\chi = 0. \quad (81)$$

In sectors with particle number  $N=0$  or  $N=n$  one can easily solve for all zero modes. For example,  $Q^\dagger$  annihilates all states in the sector with  $N=n$ , such that zero modes only need to satisfy  $Q\chi = 0$ . Because of (78), the general solution of this equation reads

$$\chi = \bar{f}(\bar{z}) g \psi^{\dagger\bar{1}} \cdots \psi^{\dagger\bar{n}} |0\rangle, \quad (82)$$

where  $\bar{f}(\bar{z})$  is some antiholomorphic function. Of course, the number of *normalizable* solutions depends on the gauge and gravitational background fields encoded in the superpotential  $g$ . With the help of the novel result (82) we shall find the explicit form of the zero modes on  $\mathbb{C}P^n$  in the following section.

## 5 The Dirac Operator on $\mathbb{C}P^n$

The ubiquitous two-dimensional  $\mathbb{C}P^n$  models possess remarkable similarities with non-Abelian gauge theories in 3+1 dimensions [23]. They are frequently used as toy models displaying interesting physics like fermion-number violation analogous to the electroweak theory [24] or spin excitations in quantum Hall systems [25]. Their instanton solutions have been studied in [26], and their  $\mathcal{N}=2$  supersymmetric extensions describe integrable systems with known scattering matrices.

It would be desirable to construct manifestly supersymmetric extensions of these models on a spatial lattice. To this end we reconsider the Dirac operator on the symmetric Kähler manifolds  $\mathbb{C}P^n$ . We shall calculate the superpotential  $g$  in (77) and the explicit zero modes of the Dirac operator.

## 5.1 Complex Projective Spaces

First we briefly recall those properties of the complex projective spaces  $\mathbb{C}P^n$  which are relevant for our purposes. The space  $\mathbb{C}P^n$  consists of complex lines in  $\mathbb{C}^{n+1}$  intersecting the origin. Its elements are identified with the following equivalence classes of points  $u = (u^0, \dots, u^n) \in \mathbb{C}^{n+1} \setminus \{0\}$ ,

$$[u] = \{v = \lambda u | \lambda \in \mathbb{C}^*\},$$

such that  $\mathbb{C}P^n$  is identified with  $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ . In each class there are elements with unit norm,  $\bar{u} \cdot u = \sum \bar{u}^j u^j = 1$ , and thus there is an equivalent characterization as a coset space of spheres,  $\mathbb{C}P^n = S^{2n+1}/S^1$ . The  $u$ 's are homogeneous coordinates of  $\mathbb{C}P^n$ . We define the  $n+1$  open sets

$$U_k = \{u \in \mathbb{C}^{n+1} | u^k \neq 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}, \quad (83)$$

the classes of which cover the projective space. The  $n+1$  maps

$$\varphi_k : \mathbb{C}^n \longrightarrow [U_k], \quad z \mapsto [z^1, \dots, 1, \dots, z^n],$$

where the  $k^{\text{th}}$  coordinate is 1, define a complex analytic structure. The line element on  $\mathbb{C}^{n+1}$ ,

$$ds^2 = \sum_{j=0}^n du^j d\bar{u}^j = du \cdot d\bar{u}, \quad (84)$$

can be restricted to  $S^{2n+1}/S^1$  and has the following representation on the  $k^{\text{th}}$  chart,

$$ds^2 = \left( \frac{\partial u}{\partial z^\mu} dz^\mu + \frac{\partial u}{\partial \bar{z}^{\bar{\mu}}} d\bar{z}^{\bar{\mu}} \right) \cdot \left( \frac{\partial \bar{u}}{\partial z^\mu} dz^\mu + \frac{\partial \bar{u}}{\partial \bar{z}^{\bar{\mu}}} d\bar{z}^{\bar{\mu}} \right).$$

We shall use the (local) coordinates

$$u = \varphi_0(z) = \frac{1}{\rho} (1, z) \in U_0, \quad \text{where} \quad \rho^2 = 1 + \bar{z} \cdot z = 1 + r^2, \quad (85)$$

for representatives with non-vanishing  $u^0$ . With these coordinates the line element takes the form

$$ds^2 = \frac{1}{\rho^2} dz \cdot d\bar{z} - \frac{1}{\rho^4} (\bar{z} \cdot dz)(z \cdot d\bar{z}), \quad (86)$$

and is derived from a Kähler potential  $K = \ln \rho^2$ . This concludes our summary of  $\mathbb{C}P^n$ .

To couple fermions to the gravitational background field we must find a complex orthonormal vielbein,  $ds^2 = e^\alpha \delta_{\alpha\bar{\alpha}} e^{\bar{\alpha}}$ , and write it as the exponential of a matrix. We obtained the following representation for the vielbein of the Fubini-Study metric (86),

$$\begin{aligned} e^\alpha &= e^\alpha_\mu dz^\mu = \rho^{-1} (\mathbb{P}^\alpha_\mu + \rho^{-1} \mathbb{Q}^\alpha_\mu) dz^\mu, \\ e_\alpha &= e^\mu_\alpha \partial_\mu = \rho (\mathbb{P}^\mu_\alpha + \rho \mathbb{Q}^\mu_\alpha) \partial_\mu. \end{aligned} \quad (87)$$

Here, we have introduced the matrices

$$\mathbb{P} = \mathbb{1} - \frac{zz^\dagger}{r^2} \quad \text{and} \quad \mathbb{Q} = \frac{zz^\dagger}{r^2}, \quad z = (z^1 \dots z^n)^T. \quad (88)$$

They satisfy

$$\mathbb{P}^2 = \mathbb{P}, \quad \mathbb{Q}^2 = \mathbb{Q}, \quad \mathbb{P}\mathbb{Q} = \mathbb{Q}\mathbb{P} = 0, \quad \mathbb{P}^\dagger = \mathbb{P}, \quad \mathbb{Q}^\dagger = \mathbb{Q}, \quad (89)$$

and hence are orthogonal projection operators. For the particular space  $\mathbb{C}P^2$ , the vielbeins are known, and can be found in [27]. These known ones are related to those in (87) by a local Lorentz transformation. We have not seen explicit formulae for the vielbeins for  $n > 2$  in the literature. Expressing the vielbeins in terms of projection operators as in (87) allows us to relate the superpotentials in different representations. From (68) and (87) we obtain the connection (1,0)-form

$$\omega_{\mu\beta}^\alpha = -\frac{\bar{z}_\mu}{\rho^2} \left( \frac{1}{2} \mathbb{P}^\alpha_\beta + \mathbb{Q}^\alpha_\beta \right) + \frac{1-\rho}{\rho r^2} \mathbb{P}^\alpha_\mu \bar{z}_\beta.$$

## 5.2 Zero Modes of the Dirac Operator

In this subsection we want to determine the zero modes of the Dirac operator  $i\not{D}$  on  $\mathbb{C}P^n$ . We use the method proposed at the end of Section 4. Actually, only for odd values of  $n$  a spin bundle  $S$  exists on  $\mathbb{C}P^n$ . We can tensor  $S$  with  $L^{k/2}$ , where  $L$  is the generating line bundle, and  $k$  takes on even values. In the language of field theory this means that we couple fermions to a U(1) gauge potential  $A$ . For even values of  $n$ , there is no spin structure, so  $S$  does not exist globally. Similarly, for odd values of  $k$ ,  $L^{k/2}$  is not defined globally. There is, however, the possibility to define a generalized spin bundle  $S_c$  which is the formal tensor product of  $S$  and  $L^{k/2}$ ,  $k$  odd [28]. Again, in the language of field theory, we couple fermions to a suitably chosen U(1) gauge potential with half-integer instanton number. In both cases, the gauge potential reads

$$A = \frac{k}{2} \bar{u} \cdot du = \frac{k}{4} (\partial - \bar{\partial}) K = g_A \partial g_A^{-1} + g_A^{\dagger-1} \bar{\partial} g_A^\dagger, \quad g_A = e^{-kK/4} = (1+r^2)^{-\frac{k}{4}}, \quad (90)$$

with corresponding field strength

$$F = dA = (\partial + \bar{\partial}) A = \frac{k}{2} \bar{\partial} \partial K. \quad (91)$$

$g_A$  is the part of the superpotential  $g$  that gives rise to the gauge connection  $A$ . It remains to determine the spin connection part  $g_\omega$  of  $g \equiv g_\omega g_A$ .

When using (87,88), the equation (68) can be written in matrix notation as  $(\omega_\mu)^\alpha_\beta = (S \partial_\mu S^{-1})^\alpha_\beta$ , where

$$S = \rho(\mathbb{P} + \rho\mathbb{Q}) \stackrel{(89)}{=} \exp(\mathbb{P} \ln \rho + \mathbb{Q} \ln \rho^2) = \exp\left((\mathbb{1} + \mathbb{Q}) \ln \rho\right). \quad (92)$$

From the matrix form of  $S$  in (92) we read off the superpotential  $g_\omega$  in the *spinor representation*,

$$g_\omega = \exp \left( \frac{1}{4} (\delta_{\bar{\alpha}\beta} + \mathbb{Q}_{\bar{\alpha}\beta}) \Gamma^{\bar{\alpha}\beta} \ln \rho \right), \quad (93)$$

where we have introduced

$$\Gamma^{\bar{\alpha}\beta} \equiv \frac{1}{2} [\Gamma^{\bar{\alpha}}, \Gamma^\beta] = 2[\psi^{\dagger\bar{\alpha}}, \psi^\beta], \quad \Gamma^{\bar{\alpha}} = 2\psi^{\dagger\bar{\alpha}}, \quad \Gamma^\beta = 2\psi^\beta. \quad (94)$$

Next, we study zero modes of  $Q$  and  $Q^\dagger$  in the gauge field background (90). In the sector of interest with  $N=n$ , the superpotential  $g_\omega$  in the spinor representation simplifies as

$$g_\omega|_{N=n} = (1+r^2)^{\frac{n+1}{4}}, \quad \text{since} \quad \Gamma^{\bar{\alpha}\beta}|_{N=n} = 2\delta^{\bar{\alpha}\beta}. \quad (95)$$

All states in the  $N=n$  sector are annihilated by  $Q^\dagger$ . Zero modes  $\chi$  satisfy in addition

$$0 = Q\chi = 2i\psi^\mu \nabla_\mu \chi = 2i\psi^\mu g \partial_\mu g^{-1} \chi, \quad g = g_A g_\omega. \quad (96)$$

Using (90) and (95) we conclude that

$$\chi = g \bar{f}(\bar{z}) \psi^{\dagger\bar{1}} \dots \psi^{\dagger\bar{n}} |0\rangle = (1+r^2)^{\frac{n+1-k}{4}} \bar{f}(\bar{z}) \psi^{\dagger\bar{1}} \dots \psi^{\dagger\bar{n}} |0\rangle, \quad (97)$$

with some antiholomorphic function  $\bar{f}$ . Normalizability of  $\chi$  will put restrictions on the admissible functions  $\bar{f}$ . Since the operators  $\bar{z}^{\bar{\mu}} \partial_{\bar{\mu}}$  (no sum) commute with  $\partial_\mu$  and with each other, we can diagonalize them simultaneously on the kernel of  $\partial_\mu$ . Thus, we are led to the following most general ansatz

$$\bar{f}_m = (\bar{z}^{\bar{1}})^{m_1} \dots (\bar{z}^{\bar{n}})^{m_n}, \quad \sum_{i=1}^n m_i = m. \quad (98)$$

There are  $\binom{m+n-1}{n-1}$  independent functions of this form. The solution  $\chi$  in (97) is square-integrable if and only if

$$\begin{aligned} \|\chi\|^2 &= \int \text{dvol} (\det h) \chi^\dagger \chi \\ &\stackrel{(97)}{\propto} \int \text{d}\Omega \int \text{d}r \, r^{2m+2n-1} (1+r^2)^{-\frac{n+k+1}{2}} < \infty, \end{aligned} \quad (99)$$

so normalizable zero modes in the  $N=n$  sector exist for

$$m = 0, 1, 2, \dots, q \equiv \frac{1}{2}(k-n-1). \quad (100)$$

Note, that  $q$  is always integer-valued, since  $k$  is odd (even) if  $n$  is even (odd). Also note, that there are no zero modes in this sector for  $k < n+1$  or equivalently  $q < 0$ . In particular, for odd  $n$  and vanishing gauge potential there are no zero modes, in agreement with [29].

For  $q \geq 0$ , the total number of zero modes in the  $N = n$  sector is

$$\sum_{m=0}^q \binom{m+n-1}{n-1} = \frac{1}{n!} (q+1)(q+2) \dots (q+n). \quad (101)$$

Similar considerations show that there are no normalizable zero modes in the  $N = 0$  sector for  $q' < 0$ , where  $q' = \frac{1}{2}(-k - n - 1)$ . For  $q' \geq 0$  there are zero modes in the  $N = 0$  sector, and their number is given by (101) with  $q$  replaced by  $q'$ .

Observe, that the states in the  $N = 0$  sector are of the same (opposite) chirality as the states in the  $N = n$  sector for even (odd)  $n$ . The contribution of the zero modes in those sectors to the index of  $i\mathcal{D}$  is given by

$$\frac{1}{n!} (q+1)(q+2) \dots (q+n), \quad q = \frac{1}{2}(k - n - 1), \quad (102)$$

for all  $q \in \mathbb{Z}$ .

On the other hand, the index theorem on  $\mathbb{C}P^n$  reads [30]

$$\text{ind } i\mathcal{D} = \int_{\mathbb{C}P^n} \text{ch}(L^{-k/2}) \hat{A}(\mathbb{C}P^n) = \frac{1}{n!} (q+1)(q+2) \dots (q+n), \quad (103)$$

where  $\text{ch}$  and  $\hat{A}$  are the Chern character and the  $\hat{A}$ -genus, respectively. Note, that this index coincides with (102). This leads us to conjecture, that for positive (negative)  $k$  *all normalizable zero modes* of the Dirac operator on the complex projective spaces  $\mathbb{C}P^n$  with Abelian gauge potential (90) reside in the sector with  $N = n$  ( $N = 0$ ) and have the form (97).

We can prove this conjecture in the particular cases  $n = 1$  and  $n = 2$ . For  $\mathbb{C}P^1$  we have constructed all zero modes. The same holds true for  $\mathbb{C}P^2$  for the following reason: Let us assume that there are zero modes in the  $N = 1$  sector. According to (48) they have opposite chirality as compared to the states in the  $N = 0$  and  $N = 2$  sectors. Hence, the index would be less than the number of zero modes in the extreme sectors. On the other hand, according to the index theorem, the index (103) is equal to this number. We conclude that there can be no zero modes in the  $N = 1$  sector.

## Conclusions

We have analyzed  $D$ -dimensional quantum mechanical systems that exhibit certain amounts of supersymmetry. Taking the Hamiltonian to be the square of the Dirac operator,  $H = -\mathcal{D}^2$ , on a curved manifold and with background gauge fields, we have constructed a set of inequivalent ‘square roots’ of  $H$ . This set includes, of course, the original Dirac operator as well as additional supercharges  $Q(I_i)$ . We have shown how these can be obtained from complex structures  $I_i$ . Therefore, the existence of a higher

amount of supersymmetry puts restrictions on the admissible geometries and gauge configurations. In even dimensions,  $\mathcal{N}=1$  gives no restrictions on the background fields, while  $\mathcal{N}=2$  requires the manifold to be Kähler and the field strength to commute with the complex structure. The  $\mathcal{N}=4$  extended supersymmetry further requires space to be hyper-Kähler and the gauge field strength to commute with all three complex structures. In four dimensions this is equivalent to the field strength being (anti-)selfdual. In 8, 12, 16, ... dimensions this requirement is much less restrictive. In 8 space dimensions,  $\mathcal{N}=8$  has only trivial solutions, namely flat space without gauge fields. Again, in 16, 24, 32, ... dimensions there are non-trivial backgrounds admitting an extended  $\mathcal{N}=8$  supersymmetry.

Our construction is similar to the one given in [9]. However, our approach has the advantage that all objects can be given a geometric interpretation, like connections, curvature etc. In addition, for backgrounds admitting extended supersymmetries (in particular  $\mathcal{N}=2$ ) we can define particle-number operators  $N$  commuting with the super-Hamiltonian (even in curved space and in the presence of gauge fields). The complex nilpotent supercharges  $Q$  and  $Q^\dagger$  act as lowering and raising operators for the number operator. The condition  $Q^2=0$  translates into the existence of a superpotential  $g$  for the (spin)connection as well as for the gauge potential.

As an application, we have deformed the Dirac operator on  $\mathbb{C}P^n$  with the help of  $g$  into its free counterpart and solved the Dirac equation, for all zero modes of  $i\not{D}$ .

As already mentioned in the introduction, particular higher dimensional quantum mechanical systems can be interpreted as supersymmetric field theoretical models on a spatial lattice. The results obtained in this paper will turn out to be very useful to construct supersymmetric sigma-models on a spatial lattice. This is work in progress, and we are confident to report on these developments in the near future.

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